Wavelet Analysis of Statistical Data on Reliability Reveals the Multifractal Nature of the Flow of Failures

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Key Words : Reliability, Multifractal, Flow of Failures, Wavelet Transform, Statistical Data, Multiplicative Process, Temporal Structure, Self-Similarity, Self-Affinity.

ABSTRACT : We introduce a new field of the multifractal applications in the reliability engineering and risk analysis. The multifractal theory is a good basis for analyzing the statistical data on reliability and safety, and provides probabilistic evidence of a multiplicative process hidden in the temporal complexity of failure sequence. We apply the wavelet transformation to a series of statistical data on reliability obtained from experiments and inspections of technical state under real service conditions. For the approbation of the technique, a computer simulation study of binomial multiplicative cascade has been done. Using the WaveLlab software carried out the wavelet analysis of the process. It provides visual evidence of the multifractal nature of the flow of failures and reveals the hierarchy that governs the relative positioning of the failures in the course of time.

Nomenclature

a	: Dilation parameter of wavelet transformation
b	: Translation parameter of wavelet transformation
C_m^r	: Binomial coefficients
F(N)	: Cumulative distribution function for the lifetime N
F(x)	: Cumulative distribution function for the interval [0, x]
f(r, m, p)	: Probability density function for the binomial distribution
M(N)	: Probabilistic measure at lifetime N
m	: Total number of repeated independent Bernoulli trials
N_0	: Size of sample
N	: Lifetime of specimen in cycles of loading
n	: Number of generations in the binary subdivision of the unit time interval T
p	: Probability of failure, or a fraction of the sample size in the left subinterval of the interval T
q	: Nonfailure probability, or a fraction of the sample size in the right subinterval of the interval T
r	: Total number of failures
T	: Unit time interval [0, 1]
t(j)	: Normalized failure time of j-th member of sample
t	: Time
$W\{x(t)\}$	: Wavelet transformation of a signal $x(t)$
x	: Subinterval of the unit time interval T

Greek Symbols

α	: Shape parameter of the Weibull distribution
Δt	: Duration of the temporal subinterval, or resolution
λ	: Scale parameter of the Weibull distribution
$\lambda(t)$	: Failure rate
μ	: Probabilistic measure
$\mu(x)$	: Probabilistic measure of the temporal subinterval, located at x for the multiplicative process
$\tau(j)$	: Failure time of j-th member of sample
$\psi(t)$	: Mother wavelet

Subscripts

max	: Maximum value

1. Introduction

The purpose of the paper is to introduce a new field of the multifractal applications in the reliability engineering and risk analysis. In particular, we do not know whether some temporal structure is hidden in an apparently disordered failure process realization. The multifractal theory is a good basis for describing the failure sequence in time and can provide a deeper understanding of the nature of failure flow.

In reliability engineering for instance, the Weibull distribution is one of the most widely used distributions because through the appropriate choice of parameters a variety of failure rate behaviors can be modeled. The two-parameter Weibull distribution assumes that the failure rate $\lambda(t)$ is in the form of a power law [1]

$$\lambda(t) = \alpha\lambda^{\alpha} t^{\alpha-1}, \tag{1}$$

for all $t \geq 0$, where α and λ are positive and are referred to as the shape and scale parameters of the distribution respectively. Such power laws, with integer or fractional exponents, are in fact endless source of self-similarity or more precisely, self-affinity. [2] These functions can be qualified as self-affine functions since their graphs are similar to themselves when transformed by anisotropic dilations.

2. Mathematical Model of Stochastic Point Process

We refer to the flow of failures as a succession of the random failure events occurring in a system during its lifetime. A mathematical construction that represents the flow of failures as a set of the random points on the time scale is referred to as a stochastic point process. [3] The point process can either be modeled as a list of impulses located at times where events occur (Fig. 1, b) or as a count process (Fig. 1, a), similar in a sense to the "devil staircase" fractal. [4-6] Such a process may be called fractal when a number of the relevant statistics of the point process exhibits scaling with related scaling exponents, indicating that the

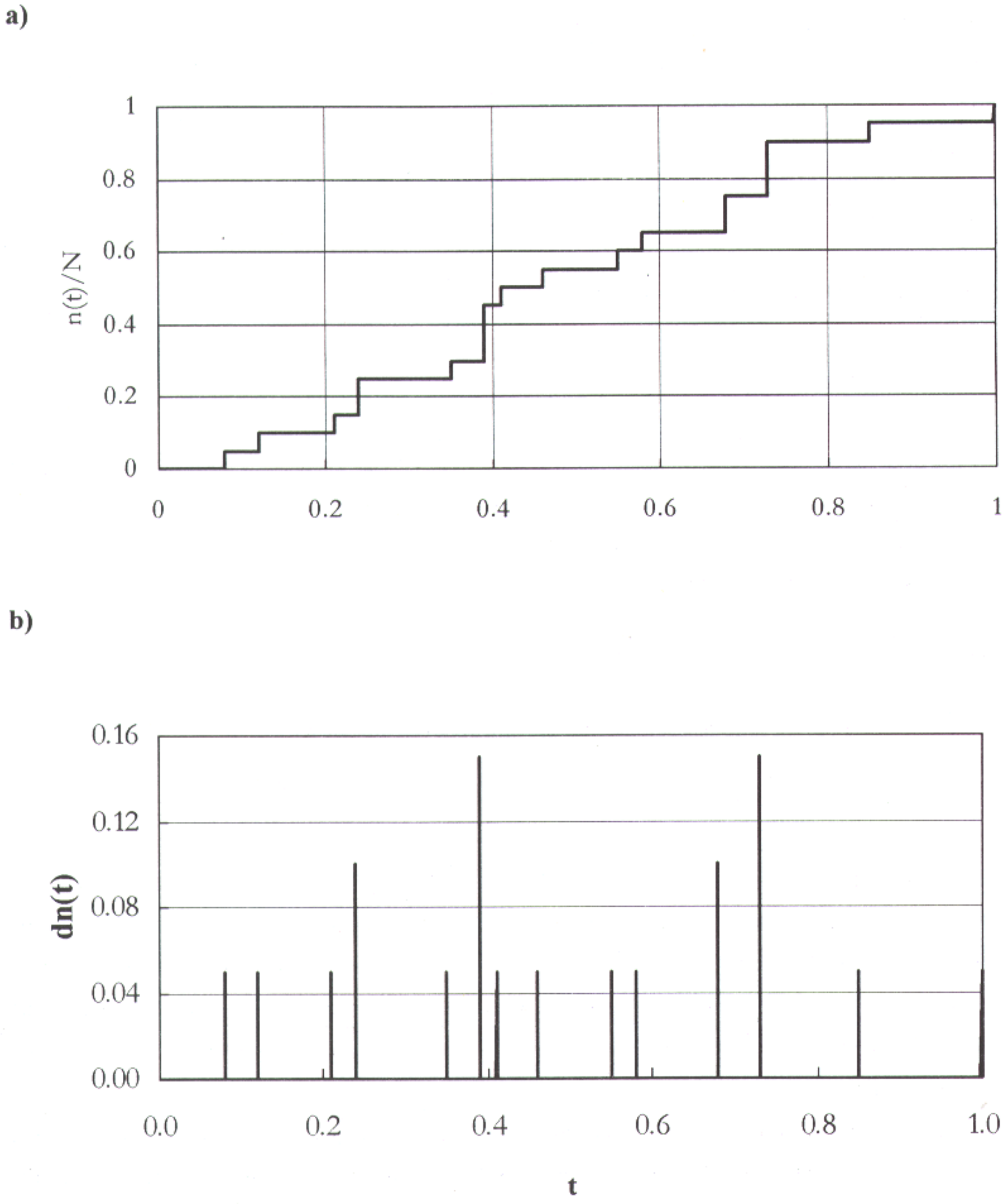


Fig.1 Two representations of a stochastic point process:
count process modeling the flow of failures (a) and sequence of idealized impulses,
occurring at normalized failure times $t(j) \in T = [0, 1]$ (b)

represented phenomenon contains clusters of points over a relatively large set of time scales. This scaling leads naturally to power-law behavior. [2]

The process realization for a sample of size N_0 is represented by a sequence of idealized impulses of vanishing width, located at specified moments of the failure time $\tau(j)$ (j = 1,..., r). Normalizing the failure time $\tau(j)$ of every j-th member of sample on the maximum value τ_{max}:

$$t(j) = \frac{\tau(j)}{\tau_{\max}}, \tag{2}$$

we can consider the failure distribution on the unit temporal interval T = [0, 1].

3. Wavelet Transformation

Wavelet transformations play a central role in the study of self-similar and self-affine signals and systems. [7-10] In terms of history, the theory of wavelet transformations dates back to the work of Grossmann and Morlet [11], and was motivated by applications in seismic data analysis.

Most generally, the wavelet transformation *W{x(t)}* of a signal *x(t)* [10]

$$W\{x(t)\} = X_b^a = \int_{-\infty}^{\infty} x(t)\,\psi_{ab}(t)\,dt, \tag{3}$$

is defined in terms of projections of *x(t)* onto a family of functions of the form

$$\psi_{ab}(t) = |a|^{-1/2}\,\psi\left(\frac{t-b}{a}\right), \tag{4}$$

normalized by $|a|^{-1/2}$. This family of functions formed by the dilations, which are controlled by the positive real number $a \in R+$, and translations which are controlled by the real number $b \in R$, of a single function $\psi(t)$ named the mother wavelet. Visually, the mother wavelet appears as a local oscillation, or soliton-like wave, in which most of the energy of the oscillation is located in a narrow region in the physical space. The dilation parameter a controls the frequency of $\psi_{ab}(t)$. The translation parameter b simply moves the wavelet throughout the domain.

The wavelet transform can be regarded as a mathematical microscope, [8, 12] for which position and magnification correspond to b and a^{-1}, respectively, and the performance of the optics is determined by the choice of the analyzing wavelet $\psi(t)$. Wavelet analysis is a powerful tool for locating singularities, [8, 12] because a singularity of a signal *x(t)* at t(j) produces a cone-like structure in the wavelet transform *W{x(t)}*, pointing towards the point a=0, b=t(j). As it was emphasized by Arneodo et al. [12], the wavelet transform assists visualization of self-similar or self-affine properties of multifractal objects. In particular, it illustrates the complexity of the multifractal under consideration, revealing the hierarchy that governs the relative positioning of the singularities of a signal *x(t)*. In the point stochastic process these singularities model the relative positioning of the failures in the course of time.

4. Computer Simulation Study

For the approbation of the technique, a computer simulation study has been done. As a first step, we have carried out a statistical multiscale analysis of m repeated independent Bernoulli trials for a sample of items of identical design and technology. In this case a binomial discrete distribution models random variable of the number r of failures in m independent Bernoulli trials, each with the probability of failure p (or nonfailure probability q). Probability density function f(r, m, p) for the binomial distribution is [1]

$$f(r,m,p) = C_m^r\, p^r\,(1-p)^{m-r}, \quad r = 0,1,2,\ldots,m\,, \tag{5}$$

where C_m^r are the binomial coefficients.

Since the work of Billingsley [13], it has been known that the binomial distribution provides an example of a probability that has a rich asymptotic structure and is, in modern terms, multifractal. [14] In fact, the binomial multifractal measure is a product of the multiplicative process (or multiplicative cascade), which attributes probabilities according to Eq. (5), to the dyadic temporal subintervals of the unit interval of time.

Distributions generated by a multiplicative process have many applications in the reliability engineering. We have considered the binomial multiplicative process, which generates a probabilistic measure on the unit temporal interval T = [0, 1]. First divide this interval T into two subintervals of equal duration $\Delta t=1/2=2^{-1}$. The left half is given a fraction p of the sample size and therefore the left subinterval has measure $\mu_0 = p$. The right-hand subinterval of time is given the remaining fraction of sample and has the probabilistic measure $\mu_1 = 1 - p = q$.

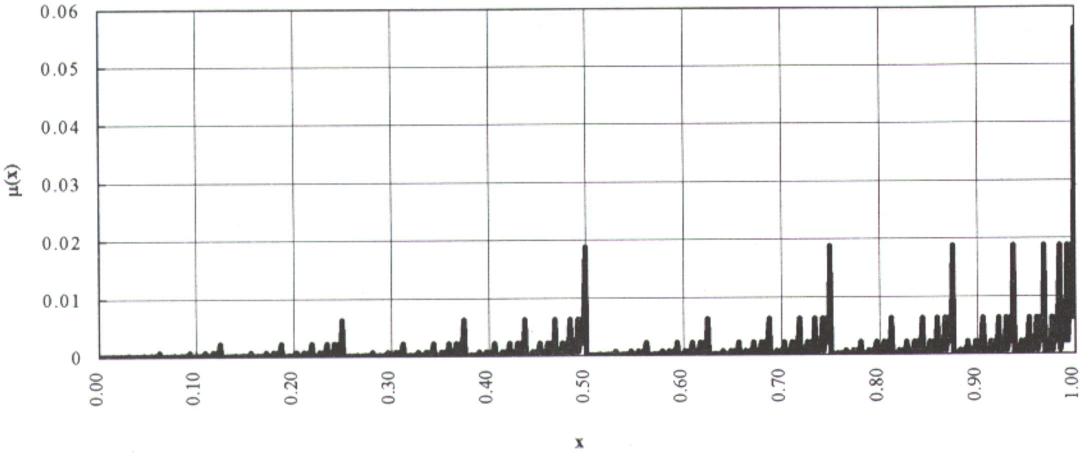
In the second generation (n=2) we increase the resolution to $\Delta t=1/4=2^{-2}$. The multiplicative process divides the population of the sample in each temporal subinterval in the same way. There are four possible outcomes, each with an associated probability, which forms the probabilistic measure

$$\mathrm{M}_2 = \{\mu_j\}_{j=0}^{2^2-1} = \{\mu_0;\, \mu_1;\, \mu_2;\, \mu_3\} = \{pp;\, pq;\, qp;\, qq\} \tag{6}$$

In the next generation (n=3) we divide each temporal subinterval into two new subintervals. A subinterval of time with the content μ_i is separated into a left-hand subinterval with measure $\mu_j = \mu_i\mu_0$ and a right-hand subinterval of time with measure $\mu_{j+1} = \mu_i\mu_1$. The whole interval of time [0, 1] is now divided into temporal subintervals of duration $\Delta t=1/8=2^{-3}$, and the set M_3 in the third generation is therefore given by the list of probabilistic measures

$$M_3 = \{\mu_j\}_{j=0}^{2^3-1} = \{ppp;\, ppq;\, pqp;\, pqq;\, qpp;\, qpq;\, qqp;\, qqq\} \tag{7}$$

This binomial multiplicative process produces shorter and shorter temporal subintervals Δt that contain less and less fractions of the total measure. Figure 2,a shows the plot of the probability mass function, that is measure $\mu(x)$ of subinterval, located at x for the binomial multiplicative process with p=0.25 and q=1-p=0.75 after n=12 generations of the multiplicative process (cascade). Figure 2,b shows the plot of the cumulative distribution function F(x), that is the measure for the interval [0, x], as a function of x a)



b)

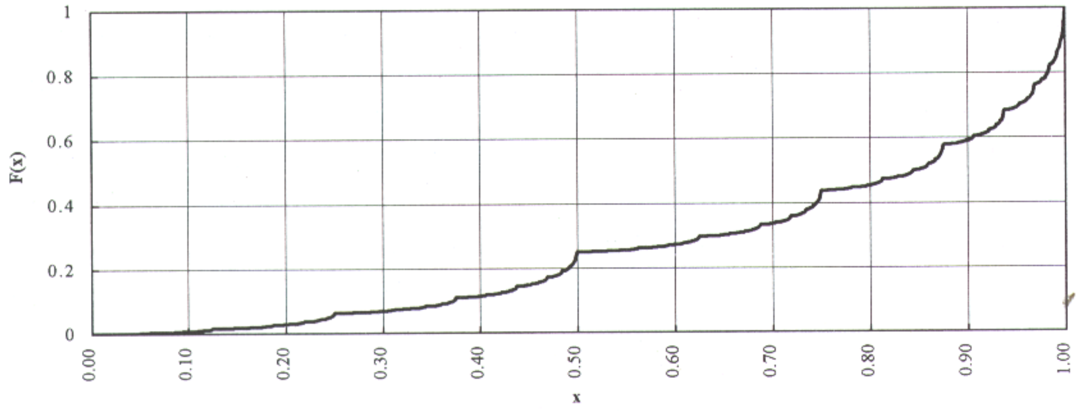


Fig. 2 The probability mass function μ(x) (a) and the cumulative distribution function F(x) (b) for the binomial multiplicative process with p=0.25

$$F(x) = \sum_{i=0}^{x} \mu_i \tag{8}$$

for the binomial multiplicative process with the same parameters.

The cumulative distribution function for the binomial multiplicative process has an evident feature, that is the self-affinity of the function F(x), describing with the following relations [6]

$$F(x) = pF(2x), \qquad for\ 0 \le x \le 1/2;$$

$$F(x) = p + (1-p)F(2x-1), \quad for\ 1/2 \le x \le 1. \tag{9}$$

So, the measure F(x) for the interval [0, x] is scaling in the sense that the left half and the right half of Fig. 2,b are obtained from the whole when transformed by anisotropic dilations.

The wavelet analysis of the multifractal measure generated by a binomial multiplicative process with parameters p=0.25 and q = 1 -- p = 0.75 was carried out by using the WaveLab package. [15] A "Sombrero" wavelet was used because it provided a marginally better visualization. The graph of the local maxima for the continuous wavelet transform of the multifractal measure generated by a binomial multiplicative process is shown in Fig. 3. It shows that the successive forkings produce a multifractal temporal structure on the unit interval.

Bernoulli trials produce the probabilistic process of failure occurrence, which results in the formation of self-similar, or rather self-affine temporal clusters. Overwhelming evidence from computer simulations indicates that these patterns are near self-affine fractals, meaning that their complication is about the same at all scales of observation. Using this classical example of repeated Bernoulli trials we have proved the technique and software. The example of computer simulation study was in detail analyzed in the articles. [16, 17]

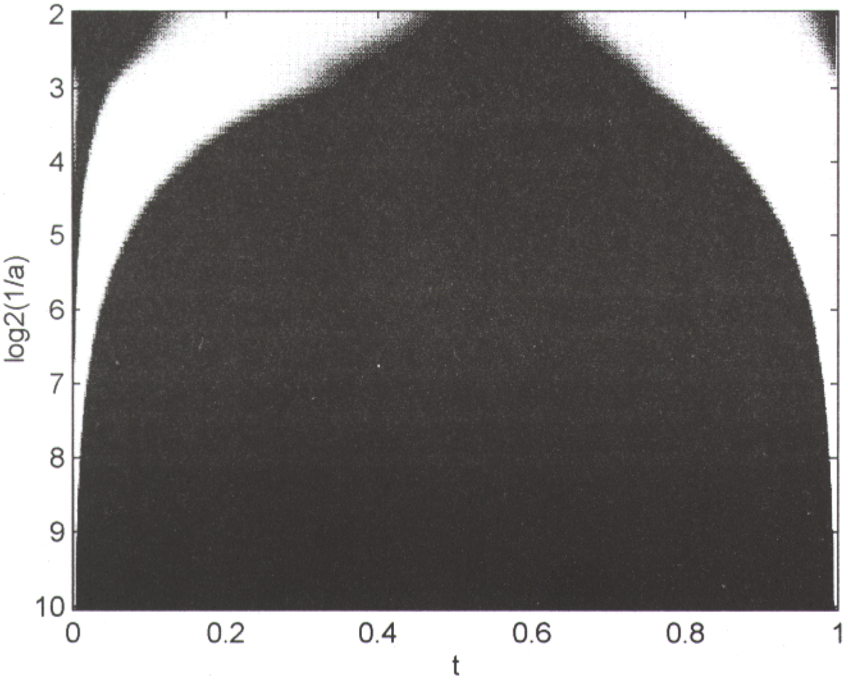


Fig. 3 The continuous wavelet transform of the multifractal measure generated by a binomial multiplicative process with p = 0.25

5. Wavelet Analysis of Statistical Data

A natural way of performing a multifractal analysis of fractal measures consists in generalizing the "classical" multifractal formalism using wavelets instead of boxes. [12]

Wavelet analysis is already known to be a powerful tool for analyzing multifractal attractors. We have applied the continuous wavelet transformation to a series of statistical data on reliability obtained from experiments and inspections of technical state under real service conditions. Wavelet analysis provides a two-dimensional unfolding of the one-dimensional realization, resolving both the time and the scale as independent variables. The wavelet transformation assists visualization of self-similar or self-affine properties of multifractal objects. In particular, it illustrates the complexity of the multifractal under consideration, revealing the hierarchy that governs the relative positioning of the failures in the course of time. The multifractal structures proposed in the flow of failures are real-time structures, in contrast to fractal attractors, which reside in phase space. Thus the wavelet transformation can be applied directly to statistical data obtained from experiments or from numerical simulations. Here we report the results of such analysis [18, 19] performed on the statistical data from experiment. [20, 21]

Birnbaum-Saunders et al. [21] data published by Bogdanoff and Kozin [20] has been used in this example. According to [20, 21] the test specimens were 6061-T6 aluminum strips, 0.061 in. thick, 4.5 in. long, and 0.5 in. wide. The specimens were cut parallel to the direction of rolling of the sheet stock. The specimens were mounted in simply supported bearings and deflected at the center with a Teflon clamp in reverse bending. The center was deflected 18 times per second and three stress amplitudes were used. These amplitudes were: 21 ksi (144.79 MPa) -- for the first sample, 26 ksi (179.26 MPa) -- for the second, and 31 ksi (213.74 MPa) -- for the third. There were 101 specimens in the first sample, 102 specimens -- in the second, and 101 specimens -- in the third. Specimens were tested to failure. Lifetimes in the unit of $N=10^3$ cycles of loading were recorded. No data were recorded at longer levels of damage.

There were an adequate number of specimens to obtain a good estimation for mean value and variance of lifetime. The plots of the cumulative (empirical) distribution function F(N) of the lifetime data for three samples of specimen as a function of the lifetime in cycles N of loading are shown in Fig. 4. They have an evident feature, that is a similarity of the empirical distribution function F(N) with the "devil staircase" prefractal. [4-6]

Using the waveLab package [15] carried out the wavelet analysis of the process. A "Sombrero" wavelet was used because it provides a marginally better visualization of the flow of failures. The graph of the local maxima for the continuous wavelet transform of the failure flow realization for the first sample of specimens is shown in Fig. 5. It shows that the successive forkings produce a multifractal temporal structure on unit interval. Increasing the magnification of the wavelet transform microscope reveals progressively the successive generations of branching.

Wavelet analysis of empirical statistical data on reliability by using the WaveLab software reveals the multifractal nature of the flow of failures, provides probabilistic evidence for the existence of a multipli-

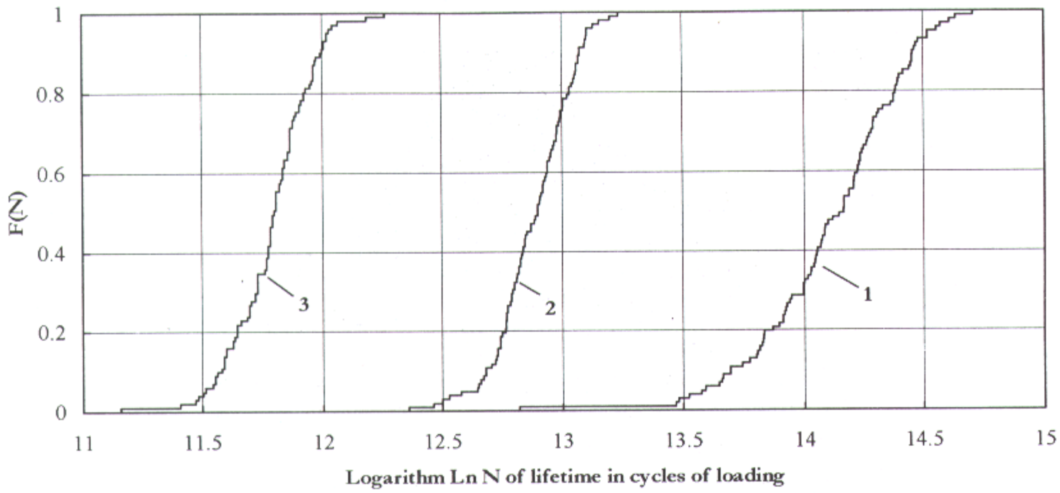


Fig. 4 Empirical distribution functions F(N) of lifetime data for the samples 1, 2 and 3 of specimens.

cative process hidden in the temporal ordering of failure sequence. This multiplicative process generates probabilistic measure, supported by a Cantor set. The ratio between the time scales of successive generation can take different values, which is another indication that a multifractal description is appropriate.

Since its implementation on a computer is not excessively time consuming and does not require enormous storage, the wavelet transform provides an efficient tool for analyzing the statistical data on reliability with multifractal structure. Its application to a variety of technical situations, such as nonparametric estimation of reliability measures, looks very useful.

6. CONCLUSIONS

We introduce here a new field of the multifractal applications in the reliability engineering and risk analysis. The multifractal theory is a good basis for describing the temporal complexity of failure sequence, for analyzing the statistical data on reliability and safety. It provides a deeper understanding the nature of the failure flow.

We have used multifractal approach to explore the temporal structure of the flow of failures. In the proposed multifractal reliability theory the emphasis switches to letting the statistical data on reliability "speak for themselves", rather than approximating the lifetime distribution by one of the parametric models.

Wavelet analysis of empirical statistical data on reliability reveals the multifractal nature of the flow of failures. Using the wavelet transform to explore the temporal structure of the flow of failures is an

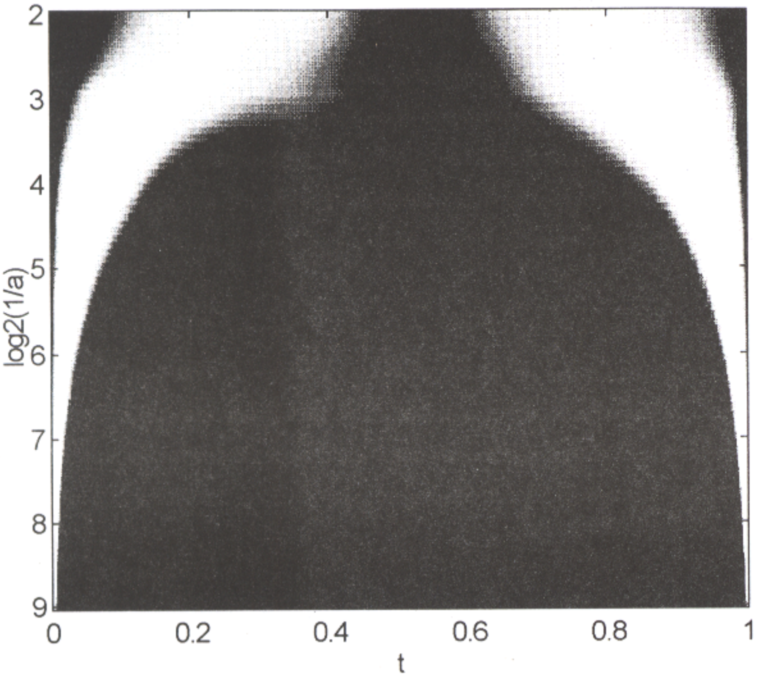


Fig. 5 The continuous wavelet transform of the failure flow realization for the first sample of specimens.

efficient tool for analyzing the statistical data on reliability. It provides probabilistic evidence for the existence of a multiplicative process hidden in the temporal complexity of failure sequence.

This is only a preliminary study: a more complete and detailed analysis should endeavor to extend this approach to the range of multiply censored statistical data on reliability and to solve an invert problem, that is recovery of the parameters of multiplicative process hidden in the flow of failures. Multifractals offer a promising approach to these problems.

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REFERENCES

(1) Lewis, E., 1987, *Introduction to Reliability Engineering*, Wiley, New York, 400 p.

(2) Schroeder, M., 1991, *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*, W.H.Freeman, New York, 430 p.

(3) Thompson, W.A.Jr., 1988, *Point Process Models with Applications to Safety and Reliability*, Chapman and Hall, New York, 146 p.

(4) Peitgen, H-O., Jurgens, H., Saupe, D., 1992, *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, New York, 984 p.

(5) Bak, P., 1986, "The Devil's staircase", *Physics Today*, Vol. 39, pp. 38 -- 45.

(6) Feder, J., 1988, *Fractals*, Plenum Press, New York and London, 284 p.

(7) Meyer, Y., 1992, *Wavelets and Operators*, Cambridge University Press, 223 p.

(8) Erlebacher, G., Hussaini, Y., Jameson, L. (eds.), 1996, *Wavelets. Theory and Applications*, Oxford University Press, New York, 510 p.

(9) Farge, M., Hunt, J., Vassilicos, J. (eds.), 1993, *Wavelets, Fractals, and Fourier Transforms*, Clarendon Press, Oxford, 403 p.

(10) Wornell, G.W., 1996, *Signal Processing with Fractals: a Wavelet-Based Approach*, Prentice Hall, 177 p.

(11) Grossmann, A., and Morlet, J., 1984, "Decompositions of Hardy Functions into Square Integrable Wavelets of Constant Shape", *SIAM J.Math.*, Vol. 15, pp. 723 -- 736.

(12) Arneodo, A., 1996, "Wavelet analysis of fractals: from the mathematical concepts to experimental reality". In: *Wavelets. theory and Applications*. G. Erlebacher, Y. Hussaini, L. Jameson (eds.). Oxford, pp. 349--502.

(13) Billingsley, P., 1965, *Ergodic Theory and Information*, Wiley, New York, p. 139.

(14) Kadanoff, L., 1990, "Scaling and Universality in Statistical Physics", *Physica*, A 163, pp. 1 -- 14.

(15) Buckheit, J., and Donoho, D., 1995, "WaveLab and Reproducible Research". In: *Wavelets and Statistics*. A. Antoniadis and G. Oppenheim (eds.). Springer, New York, pp. 55 -- 81.

(16) Zainetdinov, R.I., 1998, "Bernoulli Trials and Representation of their Results as Multifractal", *VINITI of the Russian Academy of Sciences*, Moscow, No.3902-B98, 26 p. (In Russian).

(17) Zainetdinov, R.I., 1998, "Using Wavelet Transform for Revealing the Multifractal Nature of the Sequence of Events ", *VINITI of the Russian Academy of Sciences*, Moscow, No.3903-B98, 28 p. (In Russian).

(18) Zainetdinov, R.I., 1998, "Wavelet Analysis of Statistical Data on Reliability for Exploring the Multifractal Properties of Failure-Cascading Process". In: *Wavelets and Multiscale Methods*. International Wavelets Conference "IWC -- Tangier 98". INRIA, Paris, Tangier.

(19) Zainetdinov, R.I., 1998, "Wavelet Analysis of Statistical Data on Reliability for Exploring the Multifractal Properties of the Flow of Failures", *VINITI of the Russian Academy of Sciences*, Moscow, No.3901-B98, 25 p. (In Russian).

(20) Bogdanoff, J., and Kozin, F., 1985, *Probabilistic Models of Cumulative Damage*. Wiley, New York, 342 p.

(21) Birnbaum, Z.W., Saunders, S.C., McCarty, R.C., and Elliott, R., 1950, *A Statistical Theory of Life-Length of Materials*, Boeing Airplane Company, Document No. D2-1325, Sept. 15, 1950.